Math 10B with Professor Stankova
Worksheet, Discussion #18; Tuesday, 3/20/2018
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1 Recursion Equations

1.1 Concepts

1. A first order linear recurrence equation is of the form $a_n = \alpha a_{n-1} + \beta$. The general solution to this is
$$a_n = \alpha^n a_0 + \beta \frac{\alpha^n - 1}{\alpha - 1}$$
if $\alpha \neq 1$ and $a_n = a_0 + n\beta$ otherwise.

1.2 Examples

2. Solve the recurrence relation $a_n = 2a_{n-1} + 1$ with $a_0 = 0$.

Solution: This is a first order linear recurrence equation with $\alpha = 2$ and $\beta = 1$. The general solution is then $a_n = a_0 2^n + \frac{2^n - 1}{2 - 1} = 2^n - 1$. A good thing to do now is to plug it in and verify that it is the solution. This gives
$$2^n - 1 = 2(2^{n-1} - 1) + 1 = 2 \cdot 2^{n-1} - 2 + 1 = 2^n - 1.$$

2 Differential Equations

2.1 Concepts

3. A problem of the form $y' = f(t, y)$ and $y(0) = y_0$ is called an **initial value problem (IVP)**. There is a theorem that tells us when a solution to this problem exists. It says that if f is continuous, then for every choice of y_0, the solution **exists** in a time interval $[0, T)$ for some $0 < T \leq \infty$. But, the solution may not exist everywhere and it is not guaranteed to be unique.

To solve a linear first order differential equation, first bring y, y' to one side and then divide to get the coefficient of y' to be 1 so you have something of the form $y' + P(x)y = Q(x)$. Then, multiply the integrating factor $I(x) = e^{\int P(x)dx}$. We define it this way because then $I' = I(x)P(x)$ (convince yourself of this). This gives $I(x)y' + I(x)P(x)y =$

$I(x)Q(x)$. Then the left side is just $(I(x)y)' = I(x)Q(x)$ and we can solve this by integrating then dividing by $I(x)$.

Another form of differential equation that has a nice solution is the case of separable equations. A differential equation is separable if we can write it as $y' = f(y)g(t)$, a term only involving ys and constants with a term only involving ts and constants. To do this, we write $y' = \frac{dy}{dt}$ and move all the ts to one side and the ys to the other to get $\frac{dy}{f(y)} = g(t)dt$. Now integrating gives us a solution. Often, we do not explicitly solve for y and leave them in an implicit form.

2.2 Examples

4. Find the general solution to $y' - \frac{y}{x+1} = (x+1)^2$.

Solution: The integrating factor is $I(x) = e^{\int -\frac{1}{x+1}dx} = e^{-\ln(x+1)} = \frac{1}{x+1}$. We multiply through by this to get

$$\frac{y'}{x+1} - \frac{y}{(x+1)^2} = \left(\frac{y}{x+1}\right)' = x+1.$$

Now, we integrate to get

$$\frac{y}{x+1} = \frac{(x+1)^2}{2} + C$$

and hence $y = \frac{(x+1)^3}{2} + C(x+1)$.

5. Find the solution to $y'e^y = 2t + 1$ with $y(1) = 0$.

Solution: We can rewrite this as $y' = (2t+1)e^{-y}$ and see this that is a separable equation because it is something involving t multiplied by something involving y. Now we can write this is $\frac{dy}{dt}e^y = 2t + 1$ and multiplying by dt gives

$$e^y dy = (2t+1)dt$$

and integrating gives $e^y = t^2 + t + C$ and this is the implicit solution. Now to find C, we plug in the initial condition to get $1 = e^0 = 1^2 + 1 + C = 2 + C$. Hence $C = -1$ and the solution is $e^y = t^2 + t - 1$.

2.3 Problems

6. True **FALSE** We cannot use the method of separable equations on $y' = e^{y+t}$ because it involves a sum of y and t.

Solution: We actually have $e^{y+t} = e^y e^t$ and hence it is the product of functions involving t and y and hence we can use separable equations.

7. True **FALSE** If we can use the method of separable equations, we must be able to write $y' = (ay + b)f(t)$ for a linear polynomial in terms of y.

Solution: We do not need the function in y to be linear.

8. True **FALSE** The equation $y' = y + t$ is not separable and so we do not know how to solve it.

Solution: It is indeed not separable but we can write it as $y' - y = t$ which is a form that we know how to solve with integrating factors.

9. Find the solution of $y' + \frac{y}{x} = e^x/x$ with $y(1) = 0$.

Solution: The integrating factor is $e^{\int 1/x dx} = e^{\ln x} = x$ and multiplying through gives us $xy' + y = (xy)' = e^x$. Now integrating gives $xy = e^x + C$ and hence $y = \frac{e^x}{x} + \frac{C}{x}$. Now solving for the initial condition gives us $0 = \frac{e^1}{1} + \frac{C}{1} = e + C$. Hence $C = -e$ so $y = \frac{e^x}{x} - \frac{e}{x}$.

10. Find the solution of $y' + 2xy = 2x$ with $y(0) = 0$.

Solution: The integrating factor is $e^{\int 2x dx} = e^{x^2}$. Multiplying by this gives

$$e^{x^2} y' + 2xe^{x^2} y = (e^{x^2} y)' = 2xe^{x^2}.$$

Now integrating gives $e^{x^2} y = e^{x^2} + C$ and hence $y = 1 + Ce^{-x^2}$. Plugging in the initial condition gives us that $0 = 1 + Ce^{-0} = 1 + C$ and hence $C = -1$ so the solution is $y = 1 - e^{-x^2}$.

11. Find the solution to $r' = r^2/t$ with $r(1) = 1$.

Solution: Split it to get $\frac{dr}{r^2} = \frac{dt}{t}$ and now integrating gives $-1/r = \ln t + C$ or $r = \frac{-1}{\ln t + C}$. Now we plug in the initial condition of $r(1) = 1$ and since $\ln 1 = 0$, we have that $1 = \frac{-1}{0+C} = \frac{-1}{C}$. Hence $C = -1$ and so the solution is $r(t) = \frac{-1}{-1+\ln t}$.

12. Find the general solution of $y' = 2t \sec y$.

Solution: Separating by dividing by $\sec y$ gives us $dy/\sec(y) = \cos(y)dy = 2tdt$. Now integrating gives $\sin(y) = t^2 + C$ which is the general solution.

2.4 Extra Problems

13. Find the solution to $\frac{dy}{dt} = 3t^2y^3 + e^ty^3$ with $y(0) = -1$.

Solution: We factor the right side as $y^3(t^2 + e^t)$ so we have that

$$\frac{dy}{y^3} = (3t^2 + e^t)dt.$$

Taking the integral of both sides gives

$$\frac{-1}{2y^2} = t^3 + e^t + C \implies y^2 = \frac{-1}{2t^3 + 2e^t + C}.$$

Now plugging in the initial condition that $y(0) = -1$, we have that $1 = \dfrac{-1}{0 + 2e^0 + C}$ so $2 + C = -1$ and $C = -3$. Therefore, we have

$$y^2 = \frac{-1}{2t^3 + 2e^t - 3}.$$

If we want this in terms of y only, then we have to choose which sign of y to take. We are given that $y(0) = -1$, so we take the negative square root and

$$y = -\sqrt{\frac{-1}{2t^3 + 2e^t - 3}}.$$

14. Find the solution to $\dfrac{dy}{dx} = 6xy^2$ with $y(1) = 1/4$.

Solution: Splitting gives

$$\frac{dy}{y^2} = 6xdx \implies \frac{-1}{y} = 3x^2 + C \implies y = \frac{-1}{3x^2 + C}.$$

Plugging in $y(1) = \frac{1}{4}$ gives $\frac{1}{4} = \frac{-1}{3+C}$ so $3 + C = -4$ and $C = -7$. Therefore, we have

$$y = \frac{-1}{3x^2 - 7}.$$

15. Find the solution to $\dfrac{dy}{dx} = \frac{3x^2+2x+1}{2y+1}$ with $y(0) = 1$.

Solution: Split and get

$$(2y + 1)dy = (3x^2 + 2x + 1)dx \implies y^2 + y = x^3 + x^2 + x + C.$$

We cannot explicitly solve for y easily so we have to leave it in this form. But, we can solve for c to get $y(0) = 1$ so $1 + 1 = C = 2$. Therefore, the implicit solution is

$$y^2 + y = x^3 + x^2 + x + 2.$$

16. Find the solution to $\dfrac{dy}{dt} = 2y + 3$ with $y(0) = 0$.

Solution: Separate to get

$$\frac{dy}{2y + 3} = dt \implies \frac{\ln(2y + 3)}{2} = t + C,$$

so $2y + 3 = e^{2t+C}$. We can bring the constant down as multiplying by a constant so $e^{2t+C} = e^{2t} \cdot e^C = Ce^{2t}$ and so

$$y = \frac{Ce^{2t} - 3}{2}.$$

Taking $y(0) = 0$ gives $0 = (C - 3)/2$ so $C = 3$ and we get

$$y = \frac{3e^{2t} - 3}{2}.$$

17. Find the solution to $\dfrac{dx}{dy} = e^{x-y}$ with $x(0) = 0$.

Solution: We can split e^{x-y} as $e^x \cdot e^{-y}$ and now splitting gives

$$e^{-x}dx = e^{-y}dy \implies -e^{-x} = -e^{-y} + C$$

so $e^{-x} = e^{-y} + C$ and $-x = \ln(e^{-y} + C)$ and $x = -\ln(e^{-y} + C)$. Plugging in $x(0) = 0$ gives $0 = -\ln(1 + C)$ so $1 + C = 0$ and $C = -1$. So

$$x = -\ln(e^{-y} - 1).$$